of the remaining n points on the x-axis. This yields immediately the classic interpolation formula of LaGrange.

Another solution is to determine the coefficients directly from the linear equations defining the conditions imposed. This introduces the theory of determinants, which is also suggested by the problem of indeterminate coefficients in the partial fraction expansions, and by the problem of finding the intersection of two linear graphs.

§6. Converse Problem. Determination of Roots.

The previous discussion has considered chiefly the problem, given the value of the argument x, to find the value of the rational function $f(x)$. The converse of this, given the value of $f(x)=b$, to find the value or values of x, is immediately seen to be equivalent to the problem, find the roots of the rational function $f(x)-b$, *i. e.*, the roots of the numerator when the function is written as the quotient of two polynomials. The numerical work is carried out by the Horner method of well-directed trial and error, with change of origin and synthetic substitution.

The preceding sketch is plainly rough and incomplete, but is perhaps sufficient to indicate a trend of thought which has been found to yield abundant material for a quarter's work, without sacrifice of unity of structure. Many other topics may be connected easily with those indicated if time permits.

GRAPHICAL METHODS IN TRIGONOMETRY.

By DR. L. E. DICKSON.

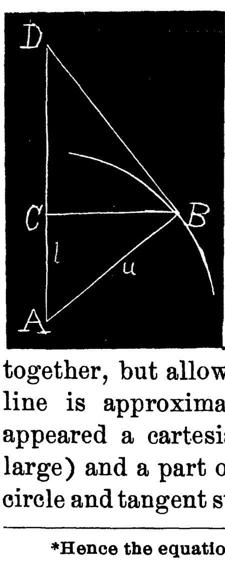
Aside from the important work on the solution of triangles by diagrams drawn to scale, graphic methods are not usually employed in trigonometry. Even if the cartesian graphs of the trigonometric functions are constructed, no serious applications are made of these graphs. They are, however, admirably adapted to the explanation of interpolation, to the visualization and retention in the memory of the ratios for the angles 0°, 90°, etc. (in contrast to their derivation as limiting values), and to the natural solution of trigonometric equations,—in particular, to the visualization of the number of angles $<180°$ having a given sine or cosine. In addition to these minor advantages resulting from a frequent appeal to the graphs, the graphic method may be employed to perform the highly important service of leading the student naturally to the majority of the fundamental trigonometric formulae, including the addition theorem and formulae for conversion of sum into product. This is in marked contrast to the current method by which each formula makes its appearance from some unseen source, to be followed by a more or less artificial proof.

An inspection of the cartesian sine curve reveals two facts (proved by recurring to the unit circle and ordinates used in constructing the graph): the symmetry of each arch and the equality of the various arches. Hence if the sine wave is moved to the right 180° and then reflected on the scale line, it coincides with its former trace as a whole;* hence $\sin(180°+\alpha)=-\sin\alpha$, for *every* angle α. Rotation of the curve about 0° through an angle 180° leads similarly to $\sin(-\alpha)=-\sin\alpha$, for every α. Reflection of the curve on the vertical through the point marked 90° leads to $\sin(180°-\alpha)=\sin\alpha$, for every α. Moving the sine wave 90° to the left, we obtain the cosine wave; hence $\sin(90°+\alpha)=\cos\alpha$, for every α. Performing the last two operations, we get $\sin\alpha=\sin(180°-\alpha)=\cos(90°-\alpha)$. Similarly, all the formulae of this type follow immediately from a combination of three of the preceding operations.

A valuable exercise is afforded by the composition of waves of different periods and phases, with emphasis on the physical applications.

As it seems preferable to *define* the cosecant, secant, and cotangent as the reciprocals of the sine, cosine, and tangent, respectively, it is desirable to construct the graphs of the former direct from the latter.

The following construction for the reciprocal $AD=r$ of a given directed line $AC=l$, the unit of length being u, is very convenient, since it yields r in the position desired for the reciprocal graph. We determine B as an intersection of a circle of center A and radius u with the perpendicular to AC at C. We make angle $ABD=90°$. Then $l:u=u:AD$.



Cartesian graphs may be converted mechanically into polar coördinate graphs. Take a rectangle $ABCD$, whose length AB is approximately π times its height BC, and make alternate forward and backward narrowfolds parallel to BC. The rectangle is compressed into a fluted surface, with D near C, and A near B. If we hold A and B together, but allow the end DC to open, we ultimately obtain a *fan*, whose outline is approximately a semi-circle.† If on the original rectangular strip appeared a cartesian sine arch with ends at A and B (the unit not being too large) and a part of the U-shaped cosecant graph, there will appear on the fan a circle and tangent straight line, representing the polar graphs of sine and cosecant.

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*Hence the equation or graph $y=\sin x$ is transformed into itself by

$$T \ : \ x'=x+180°, \ y'=-y;$$

likewise by the rotation through angle 180° about 0°, viz.,

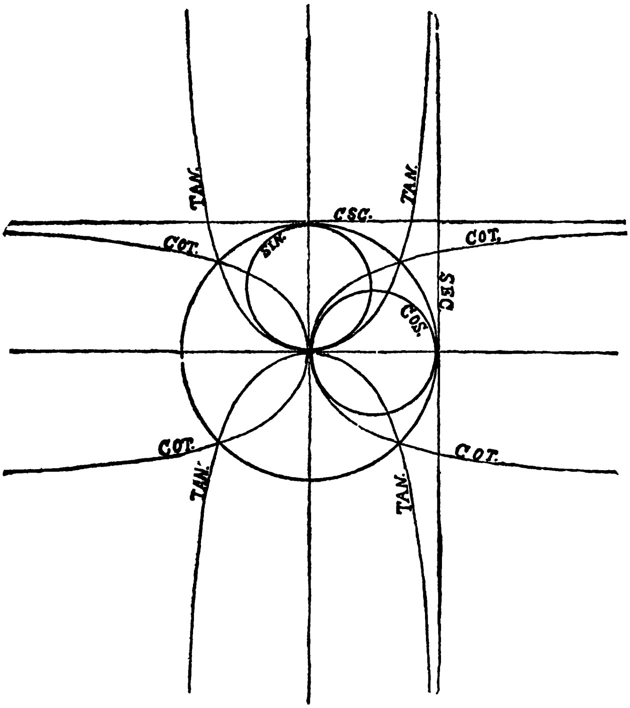
$$S \ : \ x'=-x, \ y'=-y.$$

The product $R=ST$ gives the reflection on the 90° vertical:

$$R \ : \ x'=180°-x, \ y'=y.$$

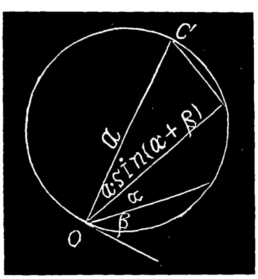
The infinite group transforming $y=\sin x$ into itself is generated by T and S.

†Let the cartesian coordinates of a point P of the rectangle be x, y, the x-axis being AB, the y-axis a perpendicular to AB at its center O. On the fan let the polar coordinates (origin O) of P be r, A. Then $r=y$, while the complement of A is measured by an arc of length x on the semi-circle.



This mechanical derivation of the polar graphs should be followed by their geometric construction on polar coördinate paper, and later by the simple formal proofs that the polar graphs of the sine and cosine are circles with respectively horizontal and vertical tangents at the origin O, and that the graphs of the cosecant and secant are straight lines tangent at the opposite end of the diameter through O. More generally, the polar graph of $r=a\sin(\alpha+\beta)$, where β is a constant angle, is the circle of diameter a, whose tangent at O makes the angle β with the initial line. The graphs of $r=a\sin\alpha$ and $r=a\cos\alpha$ are obtained by setting $\beta=0°$ and $\beta=90°$, respectively.

While the composition of waves on cartesian paper has important physical applications, the composition of the polar graphs offers greater interest as well as greater importance in the mathematical theory. We may give the following definition of composition of graphs: The points (r_1, α), (r_2, α) on the two polar graphs lead to the point (r_1+r_2, α) on the *compound* graph. The following theorem is fundamental:



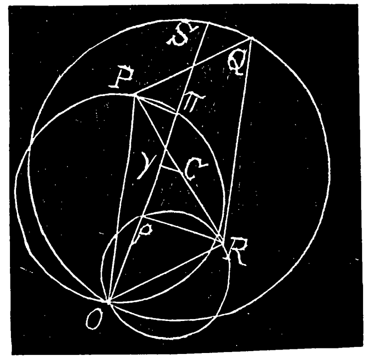
The compound of a circle on the diameter OP with a circle on the diameter OR is the circle having as diameters the diagonal OQ of the parallelogram OPQR.

Let ρ and π be the intersections of an arbitrary line through O with the given circles, and take $\pi S=O\rho$. We are to prove that S lies on the circle with center C at the middle point of PR and radius OC. Draw the perpendicular $C\gamma$ from C to OS. It is to be shown that $O\gamma=S\gamma$. This follows from $\rho\gamma=\pi\gamma$, a consequence of equal intercepts PC and CR between the parallels $P\pi$, $C\gamma$, $R\rho$.

Consider the special case in which $POR=90°$. Set $a=OP$, $b=OR$. Then the equations of the circles on OP, OR, OQ are respectively,

$$r=a\sin\alpha, \quad r=b\cos\alpha, \quad r=\sqrt{(a^2+b^2)}\sin(\alpha+\beta),$$

where $\beta = POQ$=angle between OR and the tangent at O to the circle on OQ. Hence



$$a\sin\alpha + b\cos\alpha = \sqrt{(a^2+b^2)}\sin(\alpha+\beta)$$

for every angle α. But

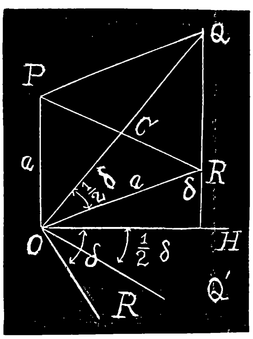
$$a = \sqrt{(a^2+b^2)}\cos\beta,\ b = \sqrt{(a^2+b^2)}\sin\beta.$$

Hence

$$\sin(\alpha+\beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta.$$

Consider next the special case $OP = OR = a$. Let OR' and OQ' be the tangents at O to the circles on the diameters OR and OQ. The initial line OH is perpendicular to OP. The circles on OP and OR are $r = a\sin\alpha$, $r = a\sin(\alpha+\delta)$, where $\delta = HOR'$. Since ORQ is isosceles, the exterior angle $\delta = 2ROQ$. Hence $\frac{1}{2}\delta = ROQ = HOQ'$ (the arms of the angles being perpendicular). Hence the compound circle on the diameter OQ is

$$r = (2a\ \cos\tfrac{1}{2}\delta)\ \sin(\alpha+\tfrac{1}{2}\delta).$$



Hence

$$\sin\alpha + \sin(\alpha+\delta) = 2\cos\ \tfrac{1}{2}\delta\ \sin(\alpha+\tfrac{1}{2}\delta).$$

Since α and δ are arbitrary angles, this formula includes the four formulae expressing $\sin\alpha \pm \sin\gamma$ and $\cos\alpha \pm \cos\gamma$ as products.

Further interesting exercises are obtained from the graphs of $r = b\sin(k\alpha+\delta)$ and their composition with $r = a\sin\alpha$.

As it is desirable to have two essentially different proofs of each theorem, the following alternative proof* of the addition theorem is suggested on account of its simplicity. Let the sum of the given angles α and β be less than 180° (the modifications of the picture for other cases being obvious). It is assumed that the sine of proportion has been previously proved.

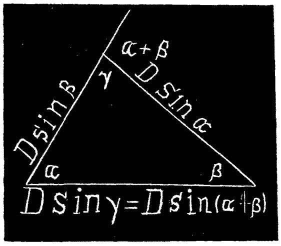
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = D \text{ (=diameter of circumscribed circle).}$$

Since the projection of one line on another equals the product of its length by the cosine of the included angle, we obtain from the figure,

$$D\ \sin(\alpha+\beta) = D\ \sin\alpha\ \cos\beta + D\ \sin\beta\ \cos\alpha.$$

*Compare Veblen's proof, MONTHLY, January, 1904, p. 7.

Likewise it seems desirable to supplement the usual analytic derivation of



$\tan\frac{1}{2}A=r/(s-a)$ by a purely geometric proof (MONTHLY, 1902, p. 36).

Theorem. *The compound of a sphere on the diameter OP with a sphere on the diameter OR is a sphere having as diameter the diagonal OQ of the parallelogram OPQR.*

The proof is similar to the above for circles. If three parallel planes make equal intercepts on one transversal they make equal intercepts on any other transversal.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

228. Proposed by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics, McKendree College, Lebanon, Ill.

Sum the infinite series

$$\frac{1}{11.13}+\frac{1}{23.25}+\frac{1}{35.37}+\frac{1}{47.49}+\frac{1}{59.61}+\cdots\cdots \qquad \text{[Oxford, 1895].}$$

Solution by the PROPOSER.

We can show that*

$$\frac{1}{2\theta}\left[\frac{1}{\theta}-\cot\theta\right]=\frac{1}{(\pi-\theta)(\pi+\theta)}+\frac{1}{(2\pi-\theta)(2\pi+\theta)}+\frac{1}{(3\pi-\theta)(3\pi+\theta)}+\cdots\cdots$$

Put $\theta=\pi/12$ and we get

$$\frac{\pi}{6}\left[\frac{12-\pi\cot 15^\circ}{\pi}\right]=\frac{144}{\pi^2}\left[\frac{1}{11.13}+\frac{1}{23.25}+\cdots\cdots\right].$$

Hence the required sum $=\dfrac{12-\pi\cot 15^\circ}{24}$.

Also solved by J. Scheffer.

229. Proposed by B. F. YANNEY, Mount Union College, Alliance, O.

If $a_1{}^n+a_2{}^n+a_3{}^n+\cdots\cdots+a_r{}^n=A^n$, $a_1{}^m+a_2{}^m+a_3{}^m+\cdots\cdots+a_r{}^m>$ or $<A^m$, according as $m<$ or $>n$; provided all the letters stand for positive real numbers.

No satisfactory solution has been received.

*Expand $\sin\theta$ in factors, take logarithms of each expression, and differentiate.